ISOCHORIC DEFORMATION HISTORIES OF FINITE COMPLEXITY

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Relations are studied which express the condition of incompressibility for deformation histories represented by a finite number of Rivlin–Ericksen tensors. In the representation of corresponding constitutive relations of differential type, the incompressibility leads to a substantial decrease in the number of independent simultaneous invariants.

Rivlin–Ericksen model[1] of the incompressible fluid,

$$\boldsymbol{\sigma} + p\boldsymbol{I} = \boldsymbol{h}_{\mathrm{M}}(\boldsymbol{E}_1, \ldots, \boldsymbol{E}_{\mathrm{M}}), \tag{1}$$

where $\boldsymbol{E}_k$'s for $k = 1, \ldots, \mathrm{M}$ are Rivlin–Ericksen kinematic tensors[1], are usually considered[2] as *a priori* definitions of a definite category of idealized "mathematical" materials. From the physical point of view this treatment possesses serious weaknesses, since "Rivlin–Ericksen materials" exhibit a physically sensible stress response only for a relatively narrow category of deformation histories. A more realistic view of rheological models of type (*1*) is offered by the concept of asymptotic representations of the general constitutive functional of an incompressible simple fluid[3]. Existing nonlinear asymptotic theories[3,4] have been limited only to a certain neighbourhood of the zero deformation history (no deformation). However, there are also other categories of deformation histories, for which an exact stress response of an incompressible simple fluid defined by Eq. (*1*) can be constructed. This category includes above all such histories, which may be represented by the Taylor expansion in the form of a finite polynomial

$$\boldsymbol{G}(s) = \sum_{k=1}^{\mathrm{N}} \frac{(-s)^k}{k!} \boldsymbol{E}_k . \tag{2}$$

We will denote such histories as histories of finite complexity N. It seems hopeful to construct the theory of Rivlin–Ericksen models of type (*1*) as a theory of asymptotic representations of the constitutive functional of an incompressible simple fluid for a certain neighbourhood of all isochoric histories with a finite complexity. Partial results of such a treatment are included in papers[5,6] dealing with nearly-viscometric flows. These possibilities motivated our interest in the structure of isochoric histories of finite complexity.

In this paper we have collected some results obtained from algebraic representations of isochoric histories of finite complexity, *i.e.* of such histories which might be represented in form (*2*) and which satisfy the isochoricity condition

$$\mathrm{Det}\,(\boldsymbol{I} + \boldsymbol{G}(s)) - 1 = 0\,. \tag{3}$$

As shown by Rivlin[7], for an N-times differentiable history at the point $s = 0$ the isochoricity condition leads to a tôtal of N scalar relations between simultaneous invariants of the N-component vector $(\boldsymbol{E}_1, \ldots, \boldsymbol{E}_N)$. A stronger assumption of the finite complexity of isochoric histories gives $3N$ algebraic relations between these invariants. Constraints imposed by these relations on the vector $(\boldsymbol{E}_1, \ldots, \boldsymbol{E}_N)$ lead to a corresponding strengthening of known algebraic representation theorems[1,8] of isotropic tensor functions of type (*1*).

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For each finite value of $\boldsymbol{G}$, the Cayley-Hamilton representation theorem allows to express relation (*3*) in the form of

$$[2\,\mathrm{tr}\,\boldsymbol{G}^3 + \mathrm{tr}^3\,\boldsymbol{G} - 3\,\mathrm{tr}\,\boldsymbol{G}.\,\mathrm{tr}\,\boldsymbol{G}^2] + 3[\mathrm{tr}^2\,\boldsymbol{G} - \mathrm{tr}\,\boldsymbol{G}^2] + 6\,\mathrm{tr}\,\boldsymbol{G} = 0\,, \tag{4}$$
$$\boldsymbol{G} = \boldsymbol{G}(s)\,, \quad s \geqq 0\,.$$

For histories of countable complexity for which expansion (*2*) for $N \to \infty$ has same meaning on a finite interval $0 \leqq s < s_0$, condition (*4*) represents the relation of the type

$$\sum_{i=1}^{\infty} a_k s^k = 0\,, \tag{5}$$

which can be satisfied only by setting

$$a_k = 0\,, \quad j = 1, 2, \ldots \tag{6}$$

Since individual expressions in relation (*4*) in the square brackets are homogeneous functions of the third, second, and first order in $\boldsymbol{G}$, the following relation is obtained for representation (2) directly from (*4*)

$$\operatorname{tr} \mathbf{E}_1 = 0 \tag{7}$$

and, by using Eq. (7), also

$$\operatorname{tr} \mathbf{E}_2 = \operatorname{tr} \mathbf{E}_1^2 . \tag{8}$$

For simplicity we introduce a notation of simultaneous invariants in the form of[1,7]

$$Z_{i_1,\ldots,i_p} = Z_{i_1,\ldots,i_p} = \operatorname{tr} (\mathbf{E}_{i_1}, \ldots, \mathbf{E}_{i_p}) \tag{9}$$

For a general $k \geqq 3$, the result of the substitution of (2) into (3) resp. (4) and expansion by (5) and (6) may be then expressed explicitly in the form of scalar relations $\mathscr{C}_{k,\infty}$ for $k \geqq 3$:

$$\mathscr{C}_{k,\infty} : \quad 0 = -\frac{Z_k}{k!} + \sum_{j=1}^{k-1} \frac{Z_{k-j,j} - Z_{k-j}Z_j}{2(k-j)!\, j!} +$$

$$+ \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} \frac{3Z_{k-j} \cdot Z_{j-i,i} - Z_{k-j}Z_{j-i}Z_i - 2Z_{k-j,j-i,i}}{6(k-j)!\,(j-i)!\, i!} . \tag{10}$$

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If nothing is assumed of $\mathbf{E}_k$ for $k > N$, the relations $\mathscr{C}_{k,\infty}$ represent a total of N scalar relations, which enable *e.g.* to express[7] invariants Z_k as polynomials in parameters $Z_{i_1,\ldots,i_p}$, $p \leqq 3$, $i_1, \ldots, i_p < k$. For histories of finite complexity N it holds

$$k > N \Rightarrow \mathbf{E}_k = 0 \tag{11}$$

Isochoric relations modified by condition (11) will be denoted by symbol $\mathscr{C}_{k,N}$ and the series

$$\mathscr{E}^N = (\mathbf{E}_1, \ldots, \mathbf{E}_N, 0, 0, \ldots) , \tag{12}$$

will be called isochorically admissible histories $\mathscr{E}^N$ of complexity N, $N = 0, 1, \ldots$, if they satisfy the isochoric relations $\mathscr{C}_{k,N}$ for $k = 1, 2, \ldots$

Some consequences can be derived for isochorically admissible histories of finite complexity from the structure of relations for $\mathscr{C}_{k,N}$.

Lemma 1. A zero history $\mathscr{E}^0$ is isochorically admissible. (It follows from the homogeneity of relations $\mathscr{C}_{k,N}$).

Lemma 2. The only isochorically admissible history of complexity 1 is the zero history $\mathscr{E}^0$.

According to the assumption, $\mathbf{E}_2 = 0$, $\mathbf{E}_3 = 0, \ldots$, from (8) it follows also $Z_{11} =$

$= \operatorname{tr} \mathbf{E}_1^2 = 0$. Since $\mathbf{E}_1$ is symmetrical, it leads necessarily to $\mathbf{E}_1 = 0$. From the foregoing, $\mathscr{E}^0$ is isochorically admissible.

Note. It follows from the theory of flows with constant stretching history[2] that nontrivial isochoric deformation histories of complexities 2, 3, 4, and ∞ can exist. It is not known whether there can exist isochoric histories of complexity N for some other finite values of N.

Lemma 3. Isochorically admissible histories $\mathscr{E}^N$ of complexity N must satisfy a total of $3N$ nontrivial scalar relations $\mathscr{C}_{k,N}$, $k = 1, \ldots, 3N$. Relation (4) is a polynomial equation of the third order in N and without the absolute term. For $\mathscr{E} = \mathscr{E}^N$, $G(s)$ is an N-th order polynomial with respect to s and relation (4) is therefore a $3N$-th order polynomial relation with respect to s. The condition of its validity on a finite interval of s leads then to the $3N$ relations of type (6).

Simultaneous invariants appear in relations for $\mathscr{C}_{k,N}$ as homogeneous additive terms of some of the following types:

$$Z_a,\ Z_{bc},\ Z_b Z_c,\ Z_d Z_{ef},\ Z_d Z_e Z_f,\ Z_{def}\,, \tag{13}$$

whose indices $(a - f)$ must for given k, N satisfy the relations

$$(a - f) \leqq N\,, \tag{14}$$

$$a = b + c = d + e + f = \mathrm{k}\,. \tag{15}$$

Consequences of these restraints can be summarized into a distribution table of parameters E_p.

With respect to the number of present E_k's, relations $\mathscr{C}_{3N,N}$, $\mathscr{C}_{3N-1,N}$ possess an equally simple structure as $\mathscr{C}_{1,N}$, $\mathscr{C}_{2,N}$, ...

TABLE I

The Distribution of $\mathbf{E}_p$'s in the Isochoric Constrains $\mathscr{C}_{k,N}$ for $N \geqq 2$

In the k^{th}-row of the Table are listed the parameters $\mathbf{E}_p$ appearing in the $\mathscr{C}_{k,N}$ for a given k and a fixed N.

k:	1	2	...	$N-1$	N	...	$2N+1$	$2N+2$	...	$3N-1$	$3N$
$\mathbf{E}_p$:	$\mathbf{E}_1$	$\mathbf{E}_1$		$\mathbf{E}_1$	$\mathbf{E}_1$		$\mathbf{E}_1$				
		$\mathbf{E}_2$		$\mathbf{E}_2$	$\mathbf{E}_2$		$\mathbf{E}_2$	$\mathbf{E}_2$			
				....	...		...	...			
				$\mathbf{E}_{N-1}$	$\mathbf{E}_{N-1}$		$\mathbf{E}_{N-1}$	$\mathbf{E}_{N-1}$		$\mathbf{E}_{N-1}$	
					$\mathbf{E}_N$		$\mathbf{E}_N$	$\mathbf{E}_N$		$\mathbf{E}_N$	$\mathbf{E}_N$

Lemma 4. For an isochorically admissible history of complexity N it holds

$$\mathrm{Det}\,(\mathbf{E}_N) = 0\,. \tag{16}$$

The relation $\mathscr{C}_{3N,N}$ may be according to Eq. (*10*) expressed in the form of

$$3Z_N\,.\,Z_{NN} - Z_N^3 - 2Z_{NNN} = 0\,,$$

which is by the Cayley–Hamilton theorem equivalent to relation (*16*).

Note. The existing method of investigation of the more detailed structure of isochoric histories of finite complexity[2] started from the canonical component representation in an orthonormal basis selected so that $\mathbf{E}_1$ has the diagonal representation. According to lemma *4* we can find such an orthonormal basis in which the component representation of $\mathbf{E}_N$ (for a flow of complexity N) yields a diagonal matrix with at most two nonzero elements.

DISCUSSION

In the study of algebraic representations of isotropic tensor functions it is sufficient to identify their argument $\mathscr{E}^N$ up to the result of a possible orthogonal transformation

$$\mathscr{E}^N \to \mathbf{Q}\mathscr{E}^N\mathbf{Q}^T = (\mathbf{Q}\mathbf{E}_1\mathbf{Q}^T, \ldots, \mathbf{Q}\mathbf{E}_N\mathbf{Q}^T, \mathbf{O}, \ldots)\,.$$

In this sense, $\mathscr{E}^N$ is fully determined by a suitable combination of $6N - 3$ independent simultaneous invariants[1]. From a formal point of view, $3N$ isochroic relations for $\mathscr{C}_{k,N}$ reduce the number of independent simultaneous invariants to $(3N - 3)$. It is not known whether this conclusion is valid for all N for which there exist nontrivial isochoric histories $\mathscr{E}^N \neq \mathscr{E}^0$ of complexity N. Nevertheless, it holds for $N = 1$ and $N = 2$.

For $N = 1$, $(3N - 3 = 0)$, according to lemma 1

$$\mathscr{E}^1 = \mathscr{E}^0\,.$$

For $N = 2$, $(3N - 3 = 3)$, we can actually find[9] such a triad of parameters, $\varkappa_1, \varkappa_2, \emptyset$

$$\varkappa_1 > 0\,, \quad \varkappa_2 \geqq 0\,, \quad 0 \leqq \emptyset \leqq 2\pi$$

through which the components of matrices $\mathbf{E}_1$ and $\mathbf{E}_2$ may be expressed in a canonical orthonormal basis:

$$\mathbf{E}_1 \sim \begin{pmatrix} \varepsilon, & \lambda_3, & \lambda_2 \\ \lambda_3, & -\varepsilon, & \lambda_1 \\ \lambda_2, & \lambda_1, & 0 \end{pmatrix}, \quad \mathbf{E}_2 \sim \begin{pmatrix} \varkappa_1 & 0 & 0 \\ 0 & \varkappa_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with

$$\lambda_1 = s\sqrt{(\varkappa_2/2)^{1/2}}, \quad \lambda_2 = c\sqrt{(\varkappa_1/2)^{1/2}},$$
$$\lambda_3 = \text{sign}(s \cdot c)\{(s^2\varkappa_1 - c^2\varkappa_2)/\sqrt{[2(s^2\varkappa_1 + c^2\varkappa_2)]^{1/2}}\},$$
$$\varepsilon = 2c \cdot s \cdot \sqrt{(\varkappa_1\varkappa_2)^{1/2}}/\sqrt{(s^2\varkappa_1 + c^2\varkappa_2)^{1/2}},$$

and where $s = \sin \emptyset$, $c = \cos \emptyset$.

It is of course impossible to express parameter ε as an analytic function of some three basic simultaneous invariants of type $Z_{i_1 \ldots i_p}$.

A special case of $\varkappa_2 = 0$, $\varkappa_1 = 2\gamma^2$ corresponds to the category of viscometric flows:

$$\mathbf{E}_1 \sim \gamma \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{E}_2 \sim 2\gamma^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The second special case, $\varkappa_1 = \varkappa_2 = \gamma^2$, *e.g.*

$$\mathbf{E}_1 \sim \gamma \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{E}_2 \sim \gamma^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

represents a category of deformation histories of complexity 2, which does not belong to motions with constant stretching history. However, a more detailed analysis[9] shows that this type of deformation histories is kinematically inadmissible.

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